Quantum effects, brane tension and large hierarchy in the brane-world

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Semiclassical Einstein's equation in five-dimension with a negative cosmological constant and conformally invariant bulk matter fields is examined in the brane world scenario with the S^1/Z_2 compactification. When numbers $N_{b,f}$ of bosonic and fermionic fields satisfy $I < \kappa^2 l^{-3}(N_b - N_f) < 32I$, we obtain an exact semiclassical solution which has two static branes with positive tension and for which the warp factor can be arbitrarily large. Here, κ^2 is the five-dimensional gravitational constant, l is a length scale determined by the negative five-dimensional cosmological constant, and I is a dimensionless positive constant of order unity. However, in order to obtain a large warp factor, fine tuning of brane tensions is required. Hence, in order to solve the hierarchy problem, we need to justify the fine tuning by another new mechanism.

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Many unified theories require spacetime dimensionality higher than four. For example, superstring theory and M-theory require ten and eleven dimensions, respectively [1]. Since our observed universe is four-dimensional, the spacetime dimensions should be reduced in such theories. One of the methods is called Kaluza-Klein compactification [2]. An alternative method was recently proposed by Randall and Sundrum [3,4], and it consists of two similar but distinct scenarios. In the first scenario [3], they considered a three-brane with negative tension as our universe and showed that the hierarchy problem may be solved by a large redshift factor, which is usually called a warp factor, between our brane and another brane called a hidden brane. In the second scenario [4], a three-brane with positive tension is considered as our universe and four-dimensional Newton's law can be realized on the brane. So far, many works were done on various aspects of these brane-world scenarios: effective four-dimensional Einstein's equation on a positive tension brane [5], weak gravity [6,7], black holes [8], inflating branes [9], cosmologies [10–14], and so on.

However, the solution of the hierarchy problem was not attempted in the second scenario while four-dimensional Newton's law was derived in the modified first scenario with radion stabilization [7]. Therefore, it might be concluded that we have to assume that our brane has negative tension in order to obtain large hierarchy. The purpose of this paper is to suggest that this may not be necessarily the case if a quantum effect called Casimir effect is taken into account. (Refs. [15,16] also discussed Casimir effect in the brane world scenario.) To be precise, we seek an exact semiclassical solution which has two static branes with positive tension and for which the warp factor can be arbitrarily large. (Refs. [17] also investigated models with two positive tension branes.) In this paper such a solution is found when numbers $N_{b,f}$ of bosonic and fermionic fields satisfy $I < \kappa^2 l^{-3}(N_b - N_f) < 32I$, where κ^2 is the five-dimensional gravitational constant, l is a length scale determined by the negative five-dimensional cosmological constant, and I is a dimensionless constant of order unity. However, in order to obtain a large warp factor, fine tuning of brane tensions is required.

This paper is organized as follows. First, we seek a static solution of semiclassical Einstein's equation including Casimir effect due to conformally invariant bulk fields. An explicit expression for the warp factor is given. Next, we extend the result to inflating branes and discuss about the cosmological constant problem in this model. Finally, we summarize this paper and discussions are given for the hierarchy problem, extension to general homogeneous, isotropic branes, and stability against perturbations.

To begin with, we consider a five-dimensional metric of the form

$$g_{MN}dx^M dx^N = e^{-2\alpha(w)}\eta_{\mu\nu}dx^\mu dx^\nu + dw^2, \tag{1}$$

where $\eta_{\mu\nu}$ represents the four-dimensional Minkowski metric. This is a general five-dimensional metric with the four-dimensional Poincaré invariance. With this metric, we analyze the semiclassical Einstein equation $G_{MN} + \Lambda g_{MN} = \kappa^2 T_{MN}$ to take into account quantum effects, where T_{MN} should be understood as an expectation value of energy momentum tensor. Hereafter, we assume that $\Lambda < 0$ as in refs. [3,4] and, for simplicity, we consider quantized conformally invariant fields as fields contributing to T_{MN}. In this case, since there is no trace anomaly in odd dimension [18], we have $T_M^M = 0$ and thus $R + 20l^{-2} = 0$, where l is defined by $\Lambda = -6l^{-2}$. This equation is written as

$$(\alpha')^2 = \frac{2}{5}\alpha'' + l^{-2}, \tag{2}$$

and solved analytically to give

$$e^{-5\alpha} = C_1 \cosh^2[\frac{5}{2} l^{-1}(w - w_0)], \quad \text{or} \tag{3}$$

$$e^{-5\alpha} = C_2 \sinh^2[\frac{5}{2} l^{-1}(w - w_0)], \quad \text{or} \tag{4}$$

$$\alpha = \pm l^{-1}(w - w_0), \tag{5}$$

where $C_{1,2}$ are positive constants. Correspondingly, energy momentum tensor should be of the following form determined by $T_{MN} = \kappa^{-2}(G_{MN} + \Lambda g_{MN})$.

$$T_{MN} dx^M dx^N = \kappa^{-2} l^{-2} \mathcal{N} e^{5\alpha(w)} (e^{-2\alpha(w)} \eta_{\mu\nu} dx^\mu dx^\nu - 4dw^2), \tag{6}$$

where $\mathcal{N} = \frac{3}{2}C_1 (> 0)$, $-\frac{3}{2}C_2 (< 0)$, and 0 for the solutions (3), (4), and (5), respectively. It is easily seen from (5) that the pure anti-deSitter spacetime used in refs. [3,4] is recovered for $\mathcal{N} = 0$.

Now let us compactify the w-direction by S^1/Z_2: we adopt the identification $w \sim w + 2L$, $w \sim -w$, where L is a constant representing the distance between two fixed points of the S^1/Z_2. This corresponds to a configuration including a 3-brane at $w = 0$ and another 3-brane at $w = L$. By denoting tensions of these branes by λ and $\bar{\lambda}$, respectively, Israel's junction conditions [19] at $w = 0$ and $w = L$ are written as

$$\lambda = 6\kappa^{-2}\alpha'(0), \quad \bar{\lambda} = -6\kappa^{-2}\alpha'(L). \tag{7}$$

For non-zero $\mathcal{N}$, α' for above solutions is independent of the absolute value of the constant $\mathcal{N}$, and these junction conditions are sufficient to determine the constant w_0 in (3) or (4) and the distance L. On the other hand, regarding an actual value of $\mathcal{N}$, because of the compactness of the w-direction and the existence of the boundaries, it is expected that $\mathcal{N} \neq 0$ in general. Actually, in ref. [15] Garriga et.al calculated energy momentum tensor for both bosonic and fermionic fields in the background (1) and showed the form (6) with $\mathcal{N}$ given by

$$\kappa^{-2} l^{-2} \mathcal{N} = \frac{1}{2} A (N_b - N_f) \left[\int_0^L e^{\alpha(w)} dw \right]^{-5}, \tag{8}$$

where A is a positive constant given by $A = \pi^2 \zeta'_R(-4)/32$ and $N_{b,f}$ are numbers of bosonic and fermionic fields, respectively. Although Garriga et.al also discussed the semiclassical Einstein equation, they gave a solution for the $\Lambda = 0$ case only, which is not relevant for the purpose of this paper. Since the coefficient $\mathcal{N}$ depends on L and w_0, the equation (8) introduces a constraint between L and w_0, or between λ and $\bar{\lambda}$ [20]. The constraint should be related to the renormalization condition adopted in ref. [15]. The results are summarized in Table I, in which $\lambda_0 \equiv \lambda/6\kappa^{-2}l^{-1}$, $\bar{\lambda}_0 \equiv \bar{\lambda}/6\kappa^{-2}l^{-1}$ and $\mathcal{A} = (5/2)(A/3)^{1/5} = O(1)$.

The so called warp factor can be defined by $\phi \equiv e^{\alpha(0)}/e^{\alpha(L)}$ and is calculated to be

$$\phi = \left[\frac{(\lambda/6\kappa^{-2}l^{-1})^2 - 1}{(\bar{\lambda}/6\kappa^{-2}l^{-1})^2 - 1} \right]^{1/5} \tag{9}$$

for $\mathcal{N} \neq 0$. (For $\mathcal{N} = 0$, we have $\phi = e^{\mp l^{-1}L}$.) Hereafter, without loss of generality, we can assume that the brane at $w = 0$ is our world. We call this brane our brane and another brane at $w = L$ a hidden brane. From (9), if we accept fine tuning of brane tensions, ϕ can be made arbitrarily large. Actually, if $N_{b,f}$ satisfy $I < \kappa^2 l^{-3}(N_b - N_f) < 32I$, where $I = \mathcal{A}^{-5}[\int_0^\infty (1 - y^2)^{-4/5} dy]^5$, then there exists a set of positive λ and positive $\bar{\lambda}$ which satisfies the constraint given in Table I and which gives an arbitrarily large warp factor. If $\kappa^2 l^{-3}(N_b - N_f) < I$ then then there exist a set of negative λ and positive $\bar{\lambda}$ which satisfies the constraint given in Table I and which gives an arbitrarily large warp factor.

We can easily extend the above results to inflating branes by simply replacing the metric $\eta_{\mu\nu}$ of the four-dimensional Minkowski metric with that of the four-dimensional deSitter spacetime $q_{\mu\nu}$. The five-dimensional metric and energy momentum tensor are given by (1) and (6) with $\eta_{\mu\nu}$ replaced by $q_{\mu\nu}$. For example, in the flat slicing,

$$q_{\mu\nu} dx^\mu dx^\nu = -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2). \tag{10}$$

Hence, by using $R + 20l^{-2} = 0$, we can show that

$$(\alpha')^2 = -\frac{2}{3} l^{-2} \mathcal{N} e^{5\alpha} + H^2 e^{2\alpha} + l^{-2}. \tag{11}$$

We shall obtain an expression for the warp factor without solving (11) explicitly. However, before doing it, let us consider the easiest case of $\mathcal{N}=0$. In this case, it is easy to integrate (11) once more to get

$$e^{-2\alpha} = l^2 H^2 \sinh^2[l^{-1}(w-w_0)]. \tag{12}$$

This is the inflating brane solution which was independently found by Nihei and Kaloper [9]. For this solution, the junction conditions at fixed points $w=0$ and $w=L$ give the following equations for w_0 and L.

$$w_0 = l\tanh^{-1}\left[\frac{6\kappa^{-2}l^{-1}}{\lambda}\right], \quad L-w_0 = l\tanh^{-1}\left[\frac{6\kappa^{-2}l^{-1}}{\bar{\lambda}}\right]. \tag{13}$$

We can now calculate the four-dimensional cosmological constants Λ_4 and $\bar{\Lambda}_4$ measured by observers on our brane and the hidden brane, respectively. These are given by

$$\Lambda_4 \equiv 3H^2 e^{2\alpha(0)}, \quad \bar{\Lambda}_4 \equiv 3H^2 e^{2\alpha(L)}, \tag{14}$$

since Hubble parameters measured by observers on these branes are $He^{\alpha(0)}$ and $He^{\alpha(L)}$, respectively. Thence, we obtain the relations $\Lambda_4 = \kappa^4\lambda^2/12 - 3l^{-2}$ and $\bar{\Lambda}_4 = \kappa^4\bar{\lambda}^2/12 - 3l^{-2}$.

Now let us consider more complicated case of $\mathcal{N}\neq 0$. In this case, one may numerically integrate (11) to obtain a solution. However, without solving it, we can obtain an analogue of the expression (9) for the warp factor. First, since (11) is a first order ordinary differential equation, its solution should have one integration constant, say w_0. Next, we can determine w_0 and the distance L between two branes by using the junction conditions (7). These are written as

$$\begin{aligned} \left(\frac{\lambda}{6\kappa^{-2}l^{-1}}\right)^2 &= -\frac{2}{3}\mathcal{N}e^{5\alpha(0)} + \frac{1}{3}\Lambda_4 l^2 + 1, \\ \left(\frac{\bar{\lambda}}{6\kappa^{-2}l^{-1}}\right)^2 &= -\frac{2}{3}\mathcal{N}e^{5\alpha(L)} + \frac{1}{3}\bar{\Lambda}_4 l^2 + 1, \end{aligned} \tag{15}$$

where Λ_4 and $\bar{\Lambda}_4$ are defined by (14). It is easily seen that for any value of H (or Λ_4) the range of allowed values of brane tensions is essentially the same as that for the flat brane case. On the contrary, an analogue of (8) should give a constraint between λ and $\bar{\lambda}$ which depends on Λ_4. Finally, we can obtain a relation between the four-dimensional cosmological constant on our brane and the warp factor. Actually, since $\bar{\Lambda}_4 = \phi^{-2}\Lambda_4$, from (15) we obtain

$$\phi = \left[\frac{(\lambda/6\kappa^{-2}l^{-1})^2 - \Lambda_4 l^2/3 - 1}{(\bar{\lambda}/6\kappa^{-2}l^{-1})^2 - \phi^{-2}\Lambda_4 l^2/3 - 1}\right]^{1/5}. \tag{16}$$

By using this expression, we can discuss the cosmological constant problem in this model. First, we can see that, if we accept a fine tuning of the tension $\bar{\lambda}$ of the hidden brane, then it is in principle possible to obtain a small Λ_4 and a large ϕ at the same time. However, because of the Λ_4-dependent constraint between brane tensions, once $\bar{\lambda}$ is fixed, λ should be determined by Λ_4. Therefore, in order to obtain a small Λ_4 and a large ϕ at the same time, we need fine tuning of tension of both branes.

Throughout this paper, semiclassical Einstein's equation with a negative cosmological constant and conformally invariant bulk matter fields has been examined in the brane world scenario with the S^1/Z_2 compactification. When numbers $N_{b,f}$ of bosonic and fermionic fields satisfy $I < \kappa^2 l^{-3}(N_b - N_f) < 32I$, we obtained an exact semiclassical solution which has two static branes with positive tension and for which the warp factor can be arbitrarily large. When $\kappa^2 l^{-3}(N_b - N_f) < I$, we have also obtained an exact semiclassical, static solution for which our brane has negative tension, the hidden brane has positive tension and the warp factor can be arbitrarily large. However, in order to obtain a large warp factor, fine tuning of brane tensions is required for both cases.

In order to justify the fine tuning of brane tensions, we need another new mechanism. In this sense, so far, the hierarchy problem has not yet been solved by this model. Besides this issue, we have another issue to be made clear in future. Namely, it is not clear whether the large warp factor is sufficient or insufficient to solve the hierarchy problem, since in this model there appears a peak of the warp factor between our brane and the hidden brane. If the peak is high enough then it may be expected that each of two branes behaves like an isolated single brane [21]. Therefore, more detailed analysis will be required to understand roles of the warp factor in this model.

Extension to general homogeneous, isotropic branes may be an interesting future work. When we set $T_{MN}=0$, a system in this category should reduce to the cosmological brane solution which was independently found by several people [12]. Since the geometry is less symmetric than the inflating brane discussed in this paper, we cannot expect the great simplification. On the contrary, since acceleration of branes changes in time, we can expect an interesting

phenomenon caused by a quantum effect called a moving mirror effect. It is well known that the moving mirror effect makes energy momentum tensor dependent of higher derivatives of boundary positions in a non-local way [18]. From results in two-dimension, it is expected that corrections due to the moving mirror effect vanish if and only if the boundary is static (the flat brane case) or uniformly accelerating (the inflating or deSitter brane case). Hence, it seems worth while investigating more general cases in detail.

Analysis of the stability against perturbations seems an important future work. In particular, the so called radion stability is worth while investigating. In ref. [15], Garriga et.al analyzed the radion stability and concluded that the $N_b > N_f$ case is unstable while the $N_b < N_f$ case is stable, implicitly assuming that the kinetic term of the radion field is given classically and that all quantum effects are included in the potential only. However, as stated above, the moving mirror effect caused by the motion of branes introduces non-local, higher-derivative corrections to the energy momentum tensor. Hence, it is expected that the kinetic term will be corrected significantly by quantum effects and that the dynamics of the radion will become non-local if the moving mirror effect is taken into account. It seems that we need more systematic analysis of the stability.

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TABLE I. Range of tensions, values of w_0 and L, and a constraint between tensions ($\mathcal{A} = (5/2)(A/3)^{1/5} = O(1)$)

N_b, N_f	$\lambda_0 \equiv \lambda/6\kappa^{-2}l^{-1}, \bar{\lambda}_0 \equiv \bar{\lambda}/6\kappa^{-2}l^{-1}$	w_0	$L - w_0$	Constraint
$N_b > N_f$	$0 < \lambda_0 \leq 1$, $0 < \bar{\lambda}_0 \leq 1$	$\frac{2}{5} l \tanh^{-1} \lambda_0$	$\frac{2}{5} l \tanh^{-1} \bar{\lambda}_0$	$\int_{-\bar{\lambda}_0}^{\lambda_0} (1 - y^2)^{-4/5} dy = \mathcal{A}[\kappa^2 l^{-3}(N_b - N_f)]^{1/5}$
	or $0 \leq -\bar{\lambda}_0 < \lambda_0 \leq 1$			
	or $0 \leq -\lambda_0 < \bar{\lambda}_0 \leq 1$			
$N_b < N_f$	$1 \leq \lambda_0 < -\bar{\lambda}_0$	$\frac{2}{5} l \tanh^{-1} \lambda_0^{-1}$	$\frac{2}{5} l \tanh^{-1} \bar{\lambda}_0^{-1}$	$\int_{\bar{\lambda}_0}^{-\lambda_0} (y^2 - 1)^{-4/5} dy = \mathcal{A}[\kappa^2 l^{-3}(N_f - N_b)]^{1/5}$
	or $1 \leq \bar{\lambda}_0 < -\lambda_0$			
$N_b = N_f$	$\lambda_0 = -\bar{\lambda}_0 = \pm 1$	Arbitrary	Arbitrary	$\lambda_0 = -\bar{\lambda}_0 = \pm 1$